# Contractibility of the orbit space of a fusion system after Steinberg

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## Background and Goal

**Theorem ([Sym98])**

*Let $G$ be a finite group and $p$ a prime dividing $|G|$. Let $|\mathcal{S}_p(G)|$ denote the $p$-subgroup complex for $G$ (considered as a topological space). Then $|\mathcal{S}_p(G)|/G$ is contractible.*

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**Theorem (Main Result)**

*Let $\mathcal{F}$ be a saturated fusion system over the finite p-group $S$. Then the space $|\mathcal{S}|/\mathcal{F}$ is contractible.*

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**Definition**

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### Definition ([Lib08])

A group $G$ is *pseudo finite at $p$* if $G$ contains a Sylow $p$-subgroup and if for every chain $(P_0 \leqslant P_1 \leqslant \cdots \leqslant P_n)$ of finite $p$-subgroups of $G$, the subgroup $N_G(P_0, P_1, \cdots, P_n) \coloneqq \bigcap_{i=0}^{n} N_G(P_i)$ has a Sylow $p$-subgroup.

# Fusion Systems

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- An *(abstract) fusion system* over a $p$-group $S$, denoted $\mathcal{F}$, is the category whose objects are the subgroups of $S$ and whose morphisms are injective group homomorphisms satisfying certain properties.
- If $\mathcal{F}$ satisfies certain condition, called *saturation axioms*, we call $\mathcal{F}$ a *saturated fusion system*. In addition, every saturated fusion system is *realizable*, which means there is some (possibly infinite) group $G$ where $\mathcal{F} = \mathcal{F}_S(G)$.

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3. $|\mathcal{S}_p(G)|/G \simeq |N/G| \simeq *$.

## Discrete Morse Theory

**Theorem ([Bro92], [For98])**

*Let $X$ be a simplicial set. Given a Morse matching for $X$, there is a quotient map $q : |X| \to Y$ of CW-complexes, where the n-cells of $Y$ are in bijection with the critical n-cells of $X$ (with characteristic maps the compositions of $q$ with the corresponding characteristic maps to $|X|$), with $q$ a homotopy equivalence.*

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- A partition of the nondegenerate simplices (called cells here) of $X$ into three types: *critical*, *redundant*, and *collapsible*.
- A bijection $c$ between the sets of redundant and collapsible cells such that, for each redundant $n$-cell $\tau$, $c(\tau)$ is an $(n+1)$-cell and an index $\iota(\tau) \in \{0, 1, \ldots, n+1\}$ such that $\tau = d_{\iota(\tau)}(c(\tau))$, where $d_i$ are the face maps of $X$.

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subject to the following condition: If $D$ is the bipartite digraph whose vertex set consists of the redundant and collapsible cells and whose edges are of the form $\tau \to c(\tau)$ for $\tau$ redundant, and $\sigma \to d_j(\sigma)$, where $j \neq \iota(c^{-1}(\tau))$, for $\sigma$ collapsible, then there is no infinite directed path in $D$.

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- **redundant** if $P_n$ is not a Sylow $p$-subgroup of $N_G(P_0, P_1, \ldots, P_n)$
- **collapsible** if $n \geqslant 1$ and $P_n$ is a Sylow $p$-subgroup of $N_G(P_0, P_1, \ldots, P_n)$

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- The index $\iota$ is given by $\iota([P_0, P_1, \ldots, P_n]) = n + 1$.

## Conclusion

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## What's next?

**Theorem ([Lin09], [Lib08])**

*Let $\mathcal{F}$ be a saturated fusion system on $S$ and let $\mathcal{S}$ be a non-empty closed $\mathcal{F}$-collection. Then $|\mathcal{S}|/\mathcal{F}$ is contractable.*

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## Fin

**Thank you for your attention!**

# Questions?