

Higher Segal conditions and localities of symmetric and alternating groups at the prime 2

The question: G finite gp, $S \in \text{Syl}_p(G)$. What is the max k s.t. $\exists g_1, \dots, g_k \in G$:

$$S \cap \bigcap_{i=1}^k S^{g_i} = 1 \text{ and } \forall 1 \leq j \leq k, S \cap \bigcap_{i \neq j} S^{g_i} \neq 1?$$

Example: $(\Sigma_7, p=2)$

$$G = \Sigma_7, S = \langle (1\ 2), (1\ 3)(2\ 4) \rangle \times \langle (5\ 6) \rangle \\ \cong D_8 \times C_2 \in \text{Syl}_2(G).$$

$$\text{Let } g_1 = (1\ 7), g_2 = (3\ 7), \text{ and } g_3 = (5\ 7).$$

$$\cdot S \cap S^{g_1} \cap S^{g_2} = \langle (5\ 6) \rangle$$

$$\cdot S \cap S^{g_1} \cap S^{g_3} = \langle (3\ 4) \rangle$$

$$\cdot S \cap S^{g_2} \cap S^{g_3} = \langle (1\ 2) \rangle$$

$$\cdot S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1 \Rightarrow k \geq 3$$

Leave space for later

Def: (Partial group - Chermak, 2013 / Thm - HL, 2025)

A part. gp L is a reduced spiny symmetric set

• Symmetric (simplicial) set: $L: \mathcal{I}^{\text{op}} \rightarrow \text{Set}$

$$\hookrightarrow \mathcal{I} = \left\{ \begin{array}{l} \text{ob: } [n] = \{0 < 1 < \dots < n\}, n \geq 0 \\ \text{morph: all functions } [n] \rightarrow [m] \end{array} \right.$$

In contrast to Δ which only has order preserving maps

• Reduced: $L_0 = *$

• Spiny: Segal maps are injective

$$L_n \hookrightarrow \underbrace{L_1 \times \dots \times L_1}_n$$

$\leadsto \Pi L_1 \xleftarrow{L_n} L_1$ defines a "product"
+ have $(\cdot)^{-1}: L_1 \rightarrow L_1$ via a full map in $\mathcal{I}$

Concretely: $\forall n \geq 1$, L_n is the set of
"multipliable" words of length n via
the product $\Pi: L_n \rightarrow L_1$ given by
 $f = [g_1 | \dots | g_n] \mapsto g_n \dots g_1$

Ex: G gp = category with one object and morphisms the elements of G
 $\Rightarrow N(G) = \text{part. gp where every word is mult.}$

Def: G gp, S set. A part. action $\cdot : G \times S \rightarrow S$ such that $\forall g, h \in G, \forall x \in S$:

① $1 \cdot x = x$

② $h \cdot (g \cdot x) \Rightarrow (hg) \cdot x$ w/ equality

③ $g \cdot x \Rightarrow g^{-1} \cdot (g \cdot x)$.

$\rightsquigarrow$ Transporter groupoid $S//G$ w/ objects S and morphisms $x \xrightarrow{g} g \cdot x$, whenever $g \cdot x$ is defined. + canonical functor $S//G \rightarrow G$.

$$\begin{aligned} & \downarrow \\ & \text{in } (N(S//G) \rightarrow N(G)) =: L_S(G) \\ & \text{"part. gp assoc. to the part. action of } G \text{ on } S." \end{aligned}$$

$\hookrightarrow$ n -simplices: $f = [g_1 | \dots | g_n] \in N(G)_n$ s.t. $\exists x \in S$ w/ $g_1 \cdot x, g_2 \cdot (g_1 \cdot x), \dots$
 $\Rightarrow f \cdot x$ is defined.

$\rightsquigarrow D(f) = \{x \in S \mid f \cdot x \text{ is defined}\}$ "domain of f "

$\rightsquigarrow$ closure space on S via $cl(A) = \bigcap_{\substack{A \subseteq D(f) \\ f \in L_n, n \geq 1}} D(f)$

Fundamental ex: G gp, $S \in \text{Syl}_p(G) \rightsquigarrow$ conjugation part action of G on $S \setminus 1$:

$g \cdot x \iff \exists x \in S \rightsquigarrow L : L_{S \setminus 1}(G)$ is the assoc. part gp

if $f = [g_1 | \dots | g_n] \in L_n$, then $D(f) = S \cap S^{g_1} \cap \dots \cap S^{g_1 \dots g_{n-1}}$

Def: (Degree of part gp - HL, 2025b) L part. gp, $\deg L = \min. k > 0$ s.t. $\forall$ words f of length $\geq k+1$, if $k+1$ of its faces are mult., then f is mult.. If no k , say $\deg = \infty$.

Prop: L part. gp, $\deg(L)=1 \iff L$ is a gp.

Pf: $\Leftarrow$ L gp means every word is mult. so implication is always true $\leadsto$ take k smallest $= 1$.

$\Rightarrow$ Induct on length: len 1 mult. by axiom. If f is len n word then by ind. def and $d_i f$ are mult. $\deg(L)=1$ means 2 faces are enough to be mult.

Example: $(\Sigma_7, p=2)$ $G = \Sigma_7$

$$S = \langle (1\ 2), (1\ 3)(2\ 4) \rangle \times \langle (5\ 6) \rangle$$

$$\cong D_8 \times C_2 \in \text{Sy}l_2(G).$$

Let $g_1 = (1\ 7)$, $g_2 = (3\ 7)$, and $g_3 = (5\ 7)$.

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$$\cdot S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1 \Rightarrow k \geq 3$$

$$L = L_{S,1}(G), f = (g_1, g_2 g_1, g_3 g_2)$$

$$\cdot f \notin L_3 \text{ b/c } D(f) = S \cap S^{g_1} \cap S^{(g_2 g_1) g_2} \cap S^{(g_3 g_2)(g_1 g_2) g_3} \\ = S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1$$

$$\cdot f_1 = d_1(f) = ((g_2 g_1) g_1, g_3 g_2) = (g_2, g_3 g_2)$$

$$\hookrightarrow D(f_1) = S \cap S^{g_2} \cap S^{(g_3 g_2) g_2} = S \cap S^{g_2} \cap S^{g_3} = \langle (1\ 2) \rangle$$

$$f_2 = d_2(f) = (g_1, (g_3 g_2)(g_2 g_1)) = (g_1, g_2 g_1)$$

$$\hookrightarrow D(f_2) = S \cap S^{g_1} \cap S^{(g_2 g_1) g_1} = S \cap S^{g_1} \cap S^{g_2} = \langle (3\ 4) \rangle$$

$$f_3 = d_3(f) = (g_1, g_2 g_1) = (g_1, g_2 g_1)$$

$$\hookrightarrow D(f_3) = S \cap S^{g_1} \cap S^{(g_2 g_1) g_1} = S \cap S^{g_1} \cap S^{g_2} = \langle (5\ 6) \rangle$$

$$\Rightarrow \deg(L) \geq 3$$

Def: (S, d) closure space.

$\mathcal{C}$ collection of closed set is Helly critical if $\bigcap \mathcal{C} = \emptyset$ and $\forall C \in \mathcal{C}, \bigcap \mathcal{C} \setminus C \neq \emptyset$

Helly number $h_d(S) =$ largest size of Helly critical.

Thm: (HL-2025b, sp. case)

G gp, $S \in \text{Sy}l_p(G)$, $L = L_{S,1}(G)$ conj part. action. $\deg(L) = h_d(S \setminus 1)$.

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$$\cdot f_1 = d_1(f) = ((g_2 g_1) g_1, g_3 g_2) = (g_2, g_3 g_2)$$

$$\hookrightarrow D(f_1) = S \cap S^{g_2} \cap S^{(g_3 g_2) g_2} = S \cap S^{g_2} \cap S^{g_3} = \langle (1\ 2) \rangle$$

$$f_2 = d_2(f) = (g_1, (g_3 g_2)(g_2 g_1)) = (g_1, g_2 g_1)$$

$$\hookrightarrow D(f_2) = S \cap S^{g_1} \cap S^{(g_2 g_1) g_1} = S \cap S^{g_1} \cap S^{g_2} = \langle (3\ 4) \rangle$$

$$S \cap S^{g_2} \cap S^{g_3} = \langle (1\ 2) \rangle$$

$$S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1 \Rightarrow k \geq 3$$

$$f_j = d_j(f) = (g_1, g_2, g_3) = (g_1, g_2, g_3)$$

$$\hookrightarrow D(f_j) = S \cap S^{g_1} \cap S^{(g_2, g_3)} = S \cap S^{g_1} \cap S^{g_2} = \langle (5\ 6) \rangle$$

$$\Rightarrow \deg(L) \geq 3$$

$$\left. \begin{aligned} C_1 &= S \cap S^{g_1} = \langle (3\ 4), (5\ 6) \rangle \\ C_2 &= S \cap S^{g_2} = \langle (1\ 2), (5\ 6) \rangle \\ C_3 &= S \cap S^{g_3} = \langle (1\ 2), (1\ 3)(2\ 4) \rangle \end{aligned} \right\} \mathcal{C} = \{C_i \mid 1 \leq i \leq 3\} \text{ is Helly critical}$$

$$\leadsto \deg(L) = h_d(S|1) \geq 3$$

Thm: (HL-to appear) $\deg(L) \leq \text{rk}_p(S)$.

Recall the p -rank is the power of p of the largest elementary abelian p -subgroup

$$\hookrightarrow C_p \times C_p \times \dots \times C_p \cong \mathbb{F}_p^n$$

My work: Compute $\deg(L)$ for $G = \Sigma_n, A_n$ & $p=2$.

notation: $n = 4m + a$.

G	$\text{rk}_2(S)$
A_n	$2m$
$\Sigma_n, a=0,1$	
$\Sigma_n, a=2,3$	$2m + 1$

Thm: (D.) $G = A_n, \Sigma_n$ $n \geq 6$ w/ $a \neq 2$ if $G = \Sigma_n, S \in \text{Sy}_2(G), L = L_{S,1}(G)$ conj part. act. Then $\deg(L) = 2m$ except when $G = \Sigma_n$ w/ $a=3$ then $\deg(L) = 2m+1$.

Note: Σ_6, Σ_7 have same 2-f.s. but diff degree of part. ^{2+3.}

Sketch of pf: A root 4-sub is a Klein 4 sub that acts regularly on its moved pts

$$C_2 \times C_2$$

$$C_2 \times C_2$$

$$\langle (1\ 2), (3\ 4) \rangle$$

$$R_1 = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$$

not root

root

Let $E = \prod R_i \cong \mathbb{F}_2^{2m}$ as a V.S. / $\mathbb{F}_2$

1-1 corresp: hyperplanes of E $\leftrightarrow$ subspaces of E^*
 (subs of index 2) of dim 1.

$$H \mapsto \varphi_H(k) = \begin{cases} 0 & \text{if } k \in H \\ 1 & \text{else} \end{cases}$$

$$\ker \varphi \longleftrightarrow \varphi$$

Lemma: basis of E^* w/ Syl int kernels = H.C. in E .

$\hookrightarrow$ this is bcz one can show $\text{Span } X = E^* \Leftrightarrow \bigcap_{f \in X} \ker(f) = 1$.

Prop: (D.) $G = A_n, \Sigma_n, n \geq 12, S, E, H \leq E$.

$a=0,1$ $\bullet$ H is a Sylow intersection (closed) $\Leftrightarrow H$ contains $m-2$ roots

$a=2,3$ $\bullet$ H is --- $\Leftrightarrow H$ contains $m-2$ or $m-1$ roots

$\leadsto$ build a basis of E^* using fct's that are supported on exactly 2 distinct blocks.

Showcase ex: $G = A_{12}, S = \bigcup_{\Sigma_k} A_{12} \quad (n=4(3)+0)$

$\langle (12), (13)(24), (15)(26)(37)(48), (910), (911)(1012) \rangle$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$E = R_1 \times R_2 \times R_3 \quad (m=3)$ basis of E^* w/ Syl int kernels: