Higher Segal Conditions and Localities of Symmetric and Alternating Groups at the Prime 2

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The Question

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- What is the maximal value of k such that there exists $g_1, \ldots, g_k \in G$ with $S \cap \bigcap_{i=1}^{k} S^{g_i} = 1$ and for all $1 \leq j \leq k$, $S \cap \bigcap_{\substack{i=1 \\ i \neq j}}^{k} S^{g_i} \neq 1$?

Σ_7 Example

Consider $G = \Sigma_7$, the symmetric group on seven symbols and let S be a Sylow 2-subgroup

$$S = \langle (1,2),\ (1,3)(2,4) \rangle \times \langle (5,6) \rangle \cong D_8 \times C_2.$$

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If $g_1 = (1,7), g_2 = (3,7)$, and $g_3 = (5,7)$, then

$$\begin{aligned} S \cap S^{g_1} \cap S^{g_2} &= \langle (5,6) \rangle \\ S \cap S^{g_1} \cap S^{g_3} &= \langle (3,4) \rangle \\ S \cap S^{g_2} \cap S^{g_3} &= \langle (1,2) \rangle \end{aligned}$$

and of course, $S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1$.

It turns out $k = 3$ is maximal for Σ_7.

Partial Groups

Definition

A *partial group* L consists of a non-empty set L_1 and for each $n \geqslant 1$, a set of multipliable words of length n, denoted L_n, via a product $\Pi : L_n \to L_1$, given by $[g_1|\cdots|g_n] \mapsto g_1 \cdots g_n$, together with an inversion $(\cdot)^{-1}$ on L_1 satisfying the following axioms:

1. Every sub-word of a multipliable word is multipliable.
2. $(\Pi : L_1 \to L_1) = \mathrm{id}_{L_1}$.
3. If $[g_1|\cdots|g_n]$ is multipliable, then so is $[g_1|\cdots|g_{i-1}|g_i \cdots g_j|g_{j+1}|\cdots|g_n]$ and their products are equal.
4. If $[g_1|\cdots|g_n]$ is multipliable, then so is $[g_n^{-1}|\cdots|g_1^{-1}|g_1|\cdots|g_n]$ and the product of the latter is 1.

Nerve

Definition

Given a category C, the *nerve* of C is the simplicial set where

- the 0-simplices, $N(C)_0 = \mathrm{Ob}(C)$,
- the 1-simplices, $N(C)_1 = \mathrm{Mor}(C)$, and
- for $n \geqslant 2$, n-simplices, $N(C)_n$, consist of n-tuples of composable morphisms of C.

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Example

A group G can be thought of as a one object category whose morphisms are elements of G. Then, $N(G)$ is a partial group where every word is multipliable.

Partial Actions

Definition

A *partial action* of a group G on a set S is a partially defined function $\cdot : G \times S \nrightarrow S$ such that for all $g, h \in G$ and $x \in S$,

1. $1 \cdot x = x$,
2. $h \cdot (g \cdot x) \implies (hg) \cdot x$ with equality if so, and
3. $g \cdot x \implies g^{-1} \cdot (g \cdot x)$.

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A partial action induces a *transporter groupoid* $S//G$ with object set S and morphisms $x \xrightarrow{g} g \cdot x$ whenever $g \cdot x$ is defined, as well as a canonical functor $S//G \to G$.

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Upon taking nerves $N(S//G) \to N(G)$, the image of this map $L_S(G)$ is the associated partial group to the partial action. The n-simplices of $L_S(G)$ are $f = [g_1|\cdots|g_n] \in N(G)_n$ for which there is $x \in S$ such that $g_1 \cdot x$ is defined, $g_2 \cdot (g_1 \cdot x)$ is defined, and so forth. In that case, we say $f \cdot x$ is defined.

Closure Operator from Partial Action

If $L = L_S(G)$ is the partial group associated to the partial action of G on S and $f \in L_n$, define the *domain* of f as the set

$$\mathbf{D}(f) = \{x \in S \mid f \cdot x\}.$$

This yields a *closure operator* on S given by

$$\mathrm{cl}(A) = \bigcap_{A \subseteq \mathbf{D}(f)} \mathbf{D}(f).$$

Conjugation Partial Action

Let G be a finite group and S a Sylow p-subgroup. Consider the conjugation partial action of G on $S \setminus 1$ given by

$$g \cdot x \iff {}^{g}x \in S,$$

and set $L = L_{S\setminus 1}(G)$ to be the associated partial group.
If $f = [g_1|\cdots|g_n] \in L_n$, then $\mathbf{D}(f) = S \cap S^{g_1} \cap \cdots \cap S^{g_n \cdots g_1}$.

Σ_7 Example (Cont.)

Let $G = \Sigma_7$ and $S = \langle (1,2),\ (1,3)(2,4) \rangle \times \langle (5,6) \rangle$ be a Sylow 2-subgroup. Consider the conjugation partial action of G on $S \setminus 1$ and set $L = L_{S \setminus 1}(G)$ the associated partial group.

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Let $g_1 = (1,7), g_2 = (3,7),\ g_3 = (5,7)$, and consider the word $f = (g_1, g_2g_1, g_3g_2)$ of length 3. Notice that $f \notin L_3$ because

$$\mathbf{D}(f) = S \cap S^{g_1} \cap S^{(g_2g_1)g_1} \cap S^{(g_3g_2)(g_2g_1)g_1} = S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1.$$

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Now set $f_1 = ((g_2g_1)g_1, g_3g_2) = (g_2, g_3g_2)$,
$f_2 = (g_1, (g_3g_2)(g_2g_1)) = (g_1, g_3g_1)$, and $f_3 = (g_1, g_2g_1)$.

$$\mathbf{D}(f_1) = S \cap S^{g_2} \cap S^{(g_3g_2)g_2} = S \cap S^{g_2} \cap S^{g_3} = \langle (1,2)\rangle$$
$$\mathbf{D}(f_2) = S \cap S^{g_1} \cap S^{(g_3g_1)g_1} = S \cap S^{g_1} \cap S^{g_3} = \langle (3,4)\rangle$$
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Degree of a Partial Group

Definition (Hackney, Lynd – 2025)

Let L be a partial group. The *degree* of L, denoted $\deg(L)$, is the smallest integer $k > 0$ such that for all words f of length $\geqslant k+1$, if $k+1$ of its faces are multipliable, then f is multipliable. If no such integer exists, we say that X has infinite degree and set $\deg(X) = \infty$.

Here, faces $d_i(f)$ of a word $f = (g_1, \dots, g_n)$ mean the following

- $d_0(f) = (g_2, \dots, g_n)$,
- $d_n(f) = (g_1, \dots, g_{n-1})$, and
- for all $0 < i < n$, $d_i(f) = (g_1, \dots, g_{i-1}, g_{i+1}g_i, g_{i+2}, \dots, g_n)$

Goal

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Proof.

If L is a group, then every word is multipliable. So, for all $n \geqslant 2$, given a word $(f_1, \ldots, f_n)$, any two faces are multipliable and we know the word itself is multipliable, so $\deg(L) = 1$.
If $\deg(L) = 1$, then let $f_1, f_2 \in L_1$ be arbitrary, consider the word $f = (f_1, f_2)$. Notice that $d_0(f) = f_2, d_2(f) = f_1 \in L_1$, hence $f \in L_2$. That is, any two elements can be multiplied; a group. □

Helly Geometry

Let X be a subset of S in the closure space (S, cl).

- The *Helly core* of X is the set

$$\mathrm{core}_{\mathrm{cl}}(X) = \bigcap_{x \in X} \mathrm{cl}\left(X \setminus \{x\}\right).$$

- X is *Helly independent* if $\mathrm{core}_{\mathrm{cl}}(X) = \{1\}$.
- The *Helly number*, denoted $h_{\mathrm{cl}}(S)$, is the largest size of a Helly independent subset.

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Although this is a notion that is only dual to our Σ_7 example, the Helly number does equal the maximal k we were looking for.

Higher Segal Degree, Helly Number, and p-rank

Theorem (Hackney, Lynd – 2025, Special Case)

Let G be a finite group, S a Sylow p-subgroup, and $L = L_{S\setminus 1}(G)$ is the partial group associated to the conjugation partial action of G on $S \setminus 1$. Then $\deg(L) = h_{\mathrm{cl}}(S \setminus 1)$.

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Theorem (Hackney, Lynd – 2026)

Under the hypotheses of the previous theorem, $\deg(L) \leq rk_p(S)$.

The Problem Statement

We would like to know how sharp the p-rank upper bound is on the degree of partial groups induced by conjugation partial actions of symmetric and alternating groups on their respective Sylow p-subgroups.

Current Results

Theorem (D.)

Let Σ_n be the symmetric group permuting n symbols and A_n the corresponding alternating group. Let $L_{S\setminus 1}(G)$ be the partial group associated to the conjugation partial action of G on a Sylow 2-subgroup S. Then

- *If $m \geqslant 3$, then $\deg(L_{S\setminus 1}(\Sigma_{2m+1})) = m = rk_2(S)$.*
- *If $m \geqslant 1$, then $\deg(L_{S\setminus 1}(A_{4m+3})) = 2m = rk_2(S)$.*

Σ_7 Example

It was no coincidence that the following intersections in Σ_7

$$S \cap S^{g_1} \cap S^{g_2} = \langle (5,6) \rangle$$
$$S \cap S^{g_1} \cap S^{g_3} = \langle (3,4) \rangle$$
$$S \cap S^{g_2} \cap S^{g_3} = \langle (1,2) \rangle$$

generate a maximal elementary abelian 2-subgroup of Σ_7, yielding a 2-rank of 3.

Work In Progress

Based on work in progress, we expect that $\deg(L_{S\setminus 1}(G)) = 2\lfloor n/4 \rfloor$ for $G = A_n$ or Σ_n and $n \geqslant 6$, unless $G = \Sigma_n$ and $n = 4m + 3$, in which case it should be one more than this.

Thank you for your attention!

Questions?