

Higher Segal Conditions and Localities of Symmetric and Alternating Groups at the Prime 2

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Representations of Finite Groups and Related Structures



The Question

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- What is the maximal value of k such that there exists $g_1, \dots, g_k \in G$ with $S \cap \bigcap_{i=1}^k S^{g_i} = 1$ and for all $1 \leq j \leq k$,

$$S \cap \bigcap_{\substack{i=1 \\ i \neq j}}^k S^{g_i} \neq 1?$$

Σ_7 Example

Consider $G = \Sigma_7$, the symmetric group on seven symbols and let S be the following Sylow 2-subgroup

$$S = \langle (1, 2), (1, 3)(2, 4) \rangle \times \langle (5, 6) \rangle \cong D_8 \times C_2.$$

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If $g_1 = (1, 7)$, $g_2 = (3, 7)$, and $g_3 = (5, 7)$, then

$$S \cap S^{g_1} \cap S^{g_2} = \langle (5, 6) \rangle$$

$$S \cap S^{g_1} \cap S^{g_3} = \langle (3, 4) \rangle$$

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and of course, $S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1$.

It turns out $k = 3$ is maximal for Σ_7 .

Partial Groups

Definition (Chermak – 2013)

A *partial group* L consists of a non-empty set L_1 and for each $n \geq 1$, a set of multipliable words of length n , denoted L_n , via a product $\Pi : L_n \rightarrow L_1$, given by $[g_1 | \cdots | g_n] \mapsto g_n \cdots g_1$, together with an inversion $(\cdot)^{-1}$ on L_1 satisfying the following axioms:

- ① Every sub-word of a multipliable word is multipliable.
- ② $(\Pi : L_1 \rightarrow L_1) = \text{id}_{L_1}$.
- ③ If $[g_1 | \cdots | g_n]$ is multipliable, then so is $[g_1 | \cdots | g_{i-1} | g_j \cdots g_i | g_{j+1} | \cdots | g_n]$ and their products are equal.
- ④ If $[g_1 | \cdots | g_n]$ is multipliable, then so is $[g_n^{-1} | \cdots | g_1^{-1} | g_1 | \cdots | g_n]$ and the product of the latter is 1.

Nerve

Definition

Given a small category C , the *nerve* of C is the simplicial set where

- the 0-simplices, $N(C)_0 = \text{Ob}(C)$,
- the 1-simplices, $N(C)_1 = \text{Mor}(C)$, and
- for $n \geq 2$, n -simplices, $N(C)_n$, consist of n -tuples of composable morphisms of C .

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Example

A group G can be thought of as a one object category whose morphisms are elements of G . Then, $N(G)$ is a partial group where every word is multipliable.

Partial Actions

Definition

A *partial action* of a group G on a set S is a partially defined function $\cdot : G \times S \rightharpoonup S$ such that for all $g, h \in G$ and $x \in S$,

- ① $1 \cdot x = x$,
- ② $h \cdot (g \cdot x) \implies (hg) \cdot x$ with equality if so, and
- ③ $g \cdot x \implies g^{-1} \cdot (g \cdot x)$.

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Upon taking nerves $N(S//G) \rightarrow N(G)$, the image of this map $L_S(G)$ is the partial group associated to the partial action. The n -simplices of $L_S(G)$ are $f = [g_1 | \cdots | g_n] \in N(G)_n$ for which there is $x \in S$ such that $g_1 \cdot x$ is defined, $g_2 \cdot (g_1 \cdot x)$ is defined, and so forth. In that case, we say $f \cdot x$ is defined.

Closure Operator from Partial Action

If $L = L_S(G)$ is the partial group associated to the partial action of G on S and $f \in L_n$, define the *domain* of f as the set

$$\mathbf{D}(f) = \{x \in S \mid f \cdot x\}.$$

This yields a *closure operator* on S given by

$$\text{cl}(A) = \bigcap_{\substack{A \subseteq \mathbf{D}(f) \\ f \in L_n, n \geq 1}} \mathbf{D}(f).$$

Conjugation Partial Action

Let G be a finite group and S a Sylow p -subgroup. Consider the conjugation partial action of G on $S \setminus 1$ given by

$$g \cdot x \iff {}^g x \in S,$$

and set $L = L_{S \setminus 1}(G)$ to be the associated partial group.

If $f = [g_1 | \cdots | g_n] \in L_n$, then $\mathbf{D}(f) = S \cap S^{g_1} \cap \cdots \cap S^{g_n \cdots g_1}$.

Σ_7 Example (Cont.)

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Let $g_1 = (1, 7)$, $g_2 = (3, 7)$, $g_3 = (5, 7)$, and consider the word $f = (g_1, g_2 g_1, g_3 g_2)$ of length 3. Notice that $f \notin L_3$ because

$$\mathbf{D}(f) = S \cap S^{g_1} \cap S^{(g_2 g_1) g_1} \cap S^{(g_3 g_2)(g_2 g_1) g_1} = S \cap S^{g_1} \cap S^{g_2} \cap S^{g_3} = 1.$$

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Now set $f_1 = ((g_2 g_1) g_1, g_3 g_2) = (g_2, g_3 g_2)$,

$f_2 = (g_1, (g_3 g_2)(g_2 g_1)) = (g_1, g_3 g_1)$, and $f_3 = (g_1, g_2 g_1)$.

$$\mathbf{D}(f_1) = S \cap S^{g_2} \cap S^{(g_3 g_2) g_2} = S \cap S^{g_2} \cap S^{g_3} = \langle (1, 2) \rangle$$

$$\mathbf{D}(f_2) = S \cap S^{g_1} \cap S^{(g_3 g_1) g_1} = S \cap S^{g_1} \cap S^{g_3} = \langle (3, 4) \rangle$$

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Degree of a Partial Group

Definition (Hackney, Lynd – 2025)

Let L be a partial group. The *degree* of L , denoted $\deg(L)$, is the smallest integer $k > 0$ such that for all words f of length $\geq k + 1$, if $k + 1$ of its faces are multipliable, then f is multipliable. If no such integer exists, we say that X has infinite degree and set $\deg(X) = \infty$.

Here, faces $d_i(f)$ of a word $f = (g_1, \dots, g_n)$ mean the following

- $d_0(f) = (g_2, \dots, g_n)$,
- $d_n(f) = (g_1, \dots, g_{n-1})$, and
- for all $0 < i < n$, $d_i(f) = (g_1, \dots, g_{i-1}, g_{i+1}g_i, g_{i+2}, \dots, g_n)$

Goal

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Proposition (Grothendieck – 1961, Special Case)

If L be a partial group, then $\deg(L) = 1 \iff L$ is a group.

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Proof.

If L is a group, then every word is multipliable. So, since the conclusion of the implication is always true, we take k to be as small as possible, that is, $\deg(L) = 1$.

If $\deg(L) = 1$, then we proceed by induction on the length of words. All words of length 1 are multipliable, and if $f = [g_1 | \cdots | g_n]$ is a word of length n , then by induction, $d_0 f$ and $d_n f$ are multipliable. Since $\deg(L) = 1$, f is multipliable. Thus, all words are multipliable which is precisely a group. $\square$

Helly Number

Let X be a subset of S in the closure space (S, cl) .

- The *Helly core* of X is the set

$$\text{core}_{\text{cl}}(X) = \bigcap_{x \in X} \text{cl}(X \setminus \{x\}).$$

- X is *Helly independent* if $\text{core}_{\text{cl}}(X) = \emptyset$.
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Example

$\{(1, 2), (3, 4), (5, 6)\}$ is Helly Independent in the Σ_7 example.

Higher Segal Degree, Helly Number, and p -rank

Theorem (Hackney, Lynd – 2025, Special Case)

Let G be a finite group, S a Sylow p -subgroup, and $L = L_{S \setminus 1}(G)$ is the partial group associated to the conjugation partial action of G on $S \setminus 1$. Then $\deg(L) = h_{\text{cl}}(S \setminus 1)$.

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Theorem (Hackney, Lynd – 2026)

Under the hypotheses of the previous theorem, $\deg(L) \leq rk_p(S)$.

The Problem Statement

We would like to compute $\deg_{S \setminus 1}(G)$ for $G = A_n, \Sigma_n$.

G	$\mathrm{rk}_p(S)$
A_n	$2 \lfloor n/4 \rfloor$
$\Sigma_n, n \equiv 0, 1 \pmod{4}$	$2 \lfloor n/4 \rfloor$
$\Sigma_n, n \equiv 2, 3 \pmod{4}$	$2 \lfloor n/4 \rfloor + 1$

Current Results

Theorem* (D.)

Let G be either Σ_n , the symmetric group permuting n symbols or A_n , the corresponding alternating group. Let $L_{S \setminus 1}(G)$ be the partial group associated to the conjugation partial action of G on a Sylow 2-subgroup S . If $n \geq 6$, then $\deg(L_{S \setminus 1}(G)) = 2 \lfloor n/4 \rfloor$. Unless $G = \Sigma_n$ and $n = 3 \pmod{4}$, in which case it is 1 more than this.

Remark

For all alternating groups and for symmetric groups such that $n \not\equiv 2 \pmod{4}$, the upper bound is sharp. However, if $n \equiv 2 \pmod{4}$, then $\deg(L_{S \setminus 1}(\Sigma_n)) = \text{rk}_2(S) - 1$.

Σ_7 Example

It was no coincidence that the following intersections in Σ_7

$$S \cap S^{g_1} \cap S^{g_2} = \langle (5, 6) \rangle$$

$$S \cap S^{g_1} \cap S^{g_3} = \langle (3, 4) \rangle$$

$$S \cap S^{g_2} \cap S^{g_3} = \langle (1, 2) \rangle$$

generate a maximal elementary abelian 2-subgroup of Σ_7 , yielding a 2-rank of 3.

Future Work

Compute degrees of symmetric and alternating groups at odd primes. Unfortunately, we don't expect the p -rank bound to be sharp.

Thank you for your attention!



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Questions?