Contractibility of the orbit space of a saturated fusion system after Steinberg

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Previous Results and Motivation

Theorem ([Sym98])

Let G be a finite group and $\mathcal{C}$ be a non-empty collection of p-subgroups of G which is closed under passage to overgroups which are p-groups and closed under conjugation. Then $N(\mathcal{C})/G$ is contractible.

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- Steinberg was able to prove the above theorem where $\mathcal{C} = \mathcal{S}_p(G) = \{P \leqslant G \mid P \text{ in a nontrivial } p\text{-subgroup of } G\}$.

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Theorem (Main Result)

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and let $\mathcal{C}$ be a closed $\mathcal{F}$-collection. Then the space $N(\mathcal{C})/\mathcal{F}$ is contractible.

Brief background on discrete Morse theory

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- A choice of index $\iota(\tau) \in \{0, 1, \ldots, n+1\}$ such that $\tau = d_{\iota(\tau)}(c(\tau))$, where d_i is the i^{th} face map.

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We say that these data are a Morse matching for X if the following condition is satisfied:

- If D is the digraph whose vertex set consists of redundant and collapsible cells and whose edges are of the form $\tau \to c(\tau)$ for a redundant cell τ and $\sigma \to d_j(\sigma)$ where $j \neq \iota(c^{-1}(\sigma))$ for a collapsible cell σ, then there is no infinite directed path.

The collapse

Theorem ([Bro92], [For98], *The Fundamental Theorem of Morse Theory*)

Let X be a simplicial set. Given a Morse matching for X, there is a quotient map $q : |X| \to Y$ of CW-complexes, where the n-cells of Y are in bijection with the critical n-cells of X (with characteristic maps the compositions of q with the corresponding characteristic maps to $|X|$), with q a homotopy equivalence.

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Definition ([Lib08])

A group G is *pseudo finite at the prime p* if G contains a Sylow p-subgroup and if for every chain $(P_0 \leqslant P_1 \leqslant \cdots \leqslant P_n)$ of finite p-subgroups of G, the subgroup $\bigcap_{i=0}^{n} N_G(P_i)$ has a Sylow p-subgroup.

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- Given an n-simplex $\gamma = (P_0, \ldots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \ldots, P_n]$ for the orbit of γ in $N(\mathcal{C})/G$.

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- Given an n-simplex $\gamma = (P_0, \ldots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \ldots, P_n]$ for the orbit of γ in $N(\mathcal{C})/G$.
- We will construct a Morse matching on $N(\mathcal{C})/G$ in 3 steps: the partition, the maps, and no infinite directed paths.

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- *redundant cell* if for all $0 \leqslant i \leqslant n$ and $Q \in \mathrm{Syl}_p(N_G(\gamma))$, QP_i is not a member of γ.
- *collapsible cell* if $n \geqslant 1$ and for some $Q \in \mathrm{Syl}_p(N_G(\gamma))$ and some $0 \leqslant i \leqslant n$, QP_i is a member of γ.

Step 2: The maps c and ι

Given a redundant cell $[\tau] = [P_0, \ldots, P_n]$, we define:

$$c([P_0, \ldots, P_n]) = \begin{cases} [P_0, P_1, \ldots, QP_i, P_{i+1}, \ldots, P_n] & \text{if } 0 \leqslant i < n, \\ [P_0, P_1, \ldots, P_n Q] & \text{if } i = n, \end{cases}$$

where $Q \in \mathrm{Syl}_p(N_G(\tau))$ and $0 \leqslant i \leqslant n$ is maximal such that $Q \nleqslant P_i$ and define $\iota([\tau]) = i + 1$. We omit the proof of bijectiveness and well-defined.

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Proposition

Given any collapsible cell $[\sigma]$, $h(d_j(\sigma)) \geqslant h(\sigma)$ for $j \neq \iota(c^{-1}([\sigma]))$. Further, equality holds if and only if $d_j([\sigma])$ is collapsible.

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$N(\mathcal{C})/G$ is contractible

Theorem ([DV24])

Let G be a pseudo finite group at the prime p and $\mathcal{C} \subseteq \mathcal{S}_p(G)$ be a collection of p-subgroups of G that is closed under conjugation and passage to overgroups in $\mathcal{S}_p(G)$. Then $N(\mathcal{C})/G$ is contractible.

Fusion Systems

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- The objects of $\mathcal{F}_S(G)$ are the subgroups of S.
- Given any two subgroups $P, Q \leqslant S$, $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_G(P, Q)$.

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Definition ([Pui06], [AKO11])

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- Given any two objects $P, Q \leqslant S$, the morphism set is such that $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$.
- Each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion.

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Definition ([Pui06], [AKO11])

- P is $\mathcal{F}$*-conjugate* to Q if they are isomorphic as objects in $\mathcal{F}$. Write $P^{\mathcal{F}}$ to denote the collection of all subgroups of S that are $\mathcal{F}$-conjugate to P.

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- If $\mathcal{F}$ satisfies further axioms so that S "behaves" like a Sylow p-subgroup of a group, $\mathcal{F}$ is said to be *saturated*.
- We say that a fusion system $\mathcal{F}$ is *realizable* if there is some (*possibly infinite*) group G with $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$.

$\mathcal{F}$ Realization Theorems

Theorem ([Rob07], Theorem 2)

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. There is some (possibly infinite) group G such that $\mathcal{F} = \mathcal{F}_S(G)$.

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Proposition ([Lib08], Proposition 3.9)

A group which realizes a saturated fusion system is pseudo finite at the prime p.

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$$(P_0, P_1, \ldots, P_n) \sim_{\mathcal{F}} (Q_0, Q_1, \ldots, Q_n)$$

if and only if there is some isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_n, Q_n)$ such that $\varphi(P_i) = Q_i$ for every $0 \leqslant i \leqslant n$.

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  if and only if there is some isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_n, Q_n)$ such that $\varphi(P_i) = Q_i$ for every $0 \leqslant i \leqslant n$.
- We denote the quotient $N(\mathcal{C})/\sim_{\mathcal{F}}$ as $N(\mathcal{C})/\mathcal{F}$.

Proving the Main Result

Theorem (Main Result)

Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S and let $\mathcal{C}$ be a closed $\mathcal{F}$-collection. Then the space $N(\mathcal{C})/\mathcal{F}$ is contractible.

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Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S and let $\mathcal{C}$ be a closed $\mathcal{F}$-collection. Then the space $N(\mathcal{C})/\mathcal{F}$ is contractible.

- By the realization theorems, we know that $\mathcal{F} = \mathcal{F}_S(G)$ where G is some pseudo finite group at the prime p and $S \in \mathrm{Syl}_p(G)$.

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- Define $\widehat{\mathcal{C}} = \bigcup_{P \in \mathcal{C}} P^G$, where $P^G = \{Q \mid Q = P^g \text{ for some } g \in G\}$.
- By construction, $\widehat{\mathcal{C}}$ is a collection of finite p-subgroups of G that is closed under conjugation and passage to overgroups in $\mathcal{S}_p(G)$.

Proving the Main Result

Proposition ([Lib08], Proposition 3.10)

Let G be a group that realizes the saturated fusion system $\mathcal{F}$ on the finite p-group S. Let $\mathcal{C}$ be an $\mathcal{F}$-collection. Then $N(\mathcal{C})/\mathcal{F} \to N(\widehat{\mathcal{C}})/G$ is an isomorphism of simplicial sets.

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- By our work with pseudo finite groups, we have that $N(\widehat{\mathcal{C}})/G$ is contractible.
- Hence, $N(\mathcal{C})/\mathcal{F}$ is contractible.

Fin

Thank you for your attention.

Questions?