Contractibility of the orbit space of a saturated fusion system after Steinberg

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Motivation

Theorem ([Sym98, Gro23])

Let G be a finite group and $\mathcal{C}$ be a non-empty collection of p-subgroups of G which is closed under passage to overgroups which are p-groups and closed under conjugation. Then $N(\mathcal{C})/G$ is contractible.

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Theorem ([Lib08, Lin09])

Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S and let $\mathcal{C}$ be a closed $\mathcal{F}$-collection. Then the space $N(\mathcal{C})/\mathcal{F}$ is contractible.

Goal

- Steinberg proves that $|\mathcal{S}_p(G)|/G$ is contractible. This is Symonds' Theorem for the case:

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- Our goal is to extend Steinberg's argument to general collections $\mathcal{C}$ and to saturated fusion systems.

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- a choice of index $\iota(\tau) \in \{0, 1, \ldots, n+1\}$ such that $\tau = d_{\iota(\tau)}(c(\tau))$, where d_i is the i-th face map.

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The collapse

Theorem ([Bro92], [For98])

Let X be a simplicial set. Given a Morse matching for X, there is a quotient map $q : |X| \to Y$ of CW-complexes, where the n-cells of Y are in bijection with the critical n-cells of X (with characteristic maps the compositions of q with the corresponding characteristic maps to $|X|$), with q a homotopy equivalence.

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Definition ([Lib08])

A group G is *pseudo finite at the prime p* if G contains a Sylow p-subgroup and if for every chain $(P_0 \leq P_1 \leq \cdots \leq P_n)$ of finite p-subgroups of G, the subgroup $\bigcap_{i=0}^{n} N_G(P_i)$ has a Sylow p-subgroup.

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- Given an n-simplex $\gamma = (P_0, \ldots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \ldots, P_n]$ for the orbit of γ.

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- Given an n-simplex $\gamma = (P_0, \ldots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \ldots, P_n]$ for the orbit of γ.
- We will construct a Morse matching on $N(\mathcal{C})/G$ in 3 steps: the partition, the maps, and no infinite directed paths.

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- a *collapsible cell* if it is not redundant and $n \geqslant 1$, and
- a *critical cell* if it is not redundant and $n = 0$.

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- $\iota([\tau]) = i + 1$ where $0 \leq i \leq n$ is maximal such that $Q \not\leq P_i$ for each $Q \in \mathrm{Syl}_p(N_G(\tau))$.
- the bijection c, after fixing a Sylow p-subgroup Q, as follows:

$$c([\tau]) = \begin{cases} [P_0, \ldots, P_{\iota([\tau])-1}, QP_{\iota([\tau])-1}, P_{\iota([\tau])}, \ldots, P_n] & \text{if } 1 \leq \iota([\tau]) \leq n, \\ [P_0, \ldots, QP_n] & \text{if } \iota([\tau]) = n+1. \end{cases}$$

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Proposition

Suppose that there is an edge in D from $[\sigma]$ to $[\gamma]$. If $[\sigma]$ is collapsible and $[\gamma]$ is redundant, then $h([\gamma]) > h([\sigma])$. Otherwise, $h([\gamma]) = h([\sigma])$.

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- Since the dimension is bounded below by 0, there are are finitely many edges in a string of collapsible cells.
- Each edge from a redundant cell to a collapsible cell is immediately preceded by an edge from a collapsible to the redundant (or the redundant cell started the path).

□

$N(\mathcal{C})/G$ is contractible

Theorem (D., Villareal)

Let G be a pseudo finite group at the prime p and $\mathcal{C} \subseteq S_p(G)$ be a collection of non-trivial p-subgroups of G that is closed under conjugation and passage to overgroups that are p-groups. The simplicial set $N(\mathcal{C})/G$ admits a Morse matching in which the set of critical cells consists of a single cell of dimension 0. Hence, $N(\mathcal{C})/G$ is contractible.

Fusion system

Definition

The *fusion system of G over $S \in \mathrm{Syl}_p(G)$* is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and has morphism sets

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) = \{c_g \mid g \in G, P^g \leq Q\}.$$

Saturated fusion system

Definition

A *satruated* fusion system over a finite p-subgroup S is the category $\mathcal{F}$ whose objects are the subgroups of S and has morphism sets

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) \subseteq \mathrm{Inj}(P,Q),$$

that satisfy additional axioms so that S “behaves” as a Sylow p-subgroup of $\mathcal{F}$.

Realizability

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Proposition ([Lib08])

A group which realizes a saturated fusion system is pseudo finite at the prime p.

$\mathcal{F}$-collections

Definition

Fix a fusion system $\mathcal{F}$ over a finite p-group S. An *$\mathcal{F}$-collection* is a union of $\mathcal{F}$-conjugacy classes of subgroups of S. An $\mathcal{F}$-collection $\mathcal{C}$ is *closed* if a subgroup $Q \leq S$ belongs to $\mathcal{C}$ whenever it contains an element of $\mathcal{C}$.

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if and only if there is some isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_n, Q_n)$ such that $\varphi(P_i) = Q_i$ for every $0 \leq i \leq n$.

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- We denote the quotient $N(\mathcal{C})/\sim_{\mathcal{F}}$ as $N(\mathcal{C})/\mathcal{F}$.

Main Result

Theorem (D., Villareal)

Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S and let $\mathcal{C}$ be a closed $\mathcal{F}$-collection. The simplicial set $N(\mathcal{C})/\mathcal{F}$ admits a Morse matching where the set of critical cells consists of one cell of dimension 0, and hence $N(\mathcal{C})/\mathcal{F}$ is contractible.

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- Define $\widehat{\mathcal{C}} = \bigcup_{P \in \mathcal{C}} P^G$, where P^G denotes the G-conjugacy class of P.
- Since $\widehat{\mathcal{C}}$ is a collection of finite p-subgroups of G that is closed under conjugation and passage to overgroups in $\mathcal{S}_p(G)$, $N(\widehat{\mathcal{C}})/G$ is contractible.

Proof II

Proposition ([Lib08])

Let G be a group that realizes the saturated fusion system $\mathcal{F}$ on the finite p-group S. Let $\mathcal{C}$ be an $\mathcal{F}$-collection. Then $N(\mathcal{C})/\mathcal{F} \to N(\widehat{\mathcal{C}})/G$ is an isomorphism of simplicial sets.

Proof II

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Hence, $N(\mathcal{C})/\mathcal{F}$ is contractible.

Thank you for your attention!

Questions?