

Contractibility of the orbit space of a saturated fusion system after Steinberg

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Motivation

Theorem ([Sym98, Gro23])

Let G be a finite group and $\mathcal{C}$ be a non-empty collection of p -subgroups of G which is closed under passage to overgroups which are p -groups and closed under conjugation. Then $N(\mathcal{C})/G$ is contractible.

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Theorem ([Lib08, Lin09])

Let $\mathcal{F}$ be a saturated fusion system over the finite p -group S and let $\mathcal{C}$ be a closed $\mathcal{F}$ -collection. Then the space $N(\mathcal{C})/\mathcal{F}$ is contractible.

Goal

- Steinberg proves that $|\mathcal{S}_p(G)|/G$ is contractible. This is Symonds' Theorem for the case:

$$\mathcal{C} = \mathcal{S}_p(G) = \{P \leq G \mid P \text{ is a nontrivial } p\text{-subgroup}\}.$$

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- Our goal is to extend Steinberg's argument to general collections $\mathcal{C}$ and to saturated fusion systems.

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- a partition of the nondegenerate simplices (or cells) into *critical* cells, *redundant* cells, and *collapsible* cells.
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- a choice of index $\iota(\tau) \in \{0, 1, \dots, n+1\}$ such that $\tau = d_{\iota(\tau)}(c(\tau))$, where d_i is the i -th face map.

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We say that these data are a *Morse matching* for X if the following condition is satisfied: if D is the digraph whose vertex set consists of redundant and collapsible cells and whose edges are of the form $\tau \rightarrow c(\tau)$ for a redundant cell τ and $\sigma \rightarrow d_j(\sigma)$ where $j \neq \iota(c^{-1}(\sigma))$ for a collapsible cell σ , then there is no infinite directed path in D .

The collapse

Theorem ([Bro92], [For98])

Let X be a simplicial set. Given a Morse matching for X , there is a quotient map $q : |X| \rightarrow Y$ of CW-complexes, where the n -cells of Y are in bijection with the critical n -cells of X (with characteristic maps the compositions of q with the corresponding characteristic maps to $|X|$), with q a homotopy equivalence.

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Definition ([Lib08])

A group G is *pseudo finite at the prime p* if G contains a Sylow p -subgroup and if for every chain $(P_0 \leq P_1 \leq \cdots \leq P_n)$ of finite p -subgroups of G , the subgroup $\bigcap_{i=0}^n N_G(P_i)$ has a Sylow p -subgroup.

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- Given an n -simplex $\gamma = (P_0, \dots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \dots, P_n]$ for the orbit of γ .

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- Given an n -simplex $\gamma = (P_0, \dots, P_n)$ in $N(\mathcal{C})$, we write $[\gamma] = [P_0, \dots, P_n]$ for the orbit of γ .
- We will construct a Morse matching on $N(\mathcal{C})/G$ in 3 steps: the partition, the maps, and no infinite directed paths.

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- a *collapsible cell* if it is not redundant and $n \geq 1$, and
- a *critical cell* if it is not redundant and $n = 0$.

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- $\iota([\tau]) = i + 1$ where $0 \leq i \leq n$ is maximal such that $Q \not\leq P_i$ for each $Q \in \text{Syl}_p(N_G(\tau))$.
- the bijection c , after fixing a Sylow p -subgroup Q , as follows:

$$c([\tau]) = \begin{cases} [P_0, \dots, P_{\iota([\tau])-1}, QP_{\iota([\tau])-1}, P_{\iota([\tau])}, \dots, P_n] & \text{if } 1 \leq \iota([\tau]) \leq n, \\ [P_0, \dots, QP_n] & \text{if } \iota([\tau]) = n + 1. \end{cases}$$

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Proposition

Suppose that there is an edge in D from $[\sigma]$ to $[\gamma]$. If $[\sigma]$ is collapsible and $[\gamma]$ is redundant, then $h([\gamma]) > h([\sigma])$. Otherwise, $h([\gamma]) = h([\sigma])$.

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- Since the dimension is bounded below by 0, there are finitely many edges in a string of collapsible cells.
- Each edge from a redundant cell to a collapsible cell is immediately preceded by an edge from a collapsible to the redundant (or the redundant cell started the path).



$N(\mathcal{C})/G$ is contractible

Theorem (D., Villareal)

Let G be a pseudo finite group at the prime p and $\mathcal{C} \subseteq S_p(G)$ be a collection of non-trivial p -subgroups of G that is closed under conjugation and passage to overgroups that are p -groups. The simplicial set $N(\mathcal{C})/G$ admits a Morse matching in which the set of critical cells consists of a single cell of dimension 0. Hence, $N(\mathcal{C})/G$ is contractible.

Fusion system

Definition

The *fusion system* of G over $S \in \text{Syl}_p(G)$ is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and has morphism sets

$$\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q) = \{c_g \mid g \in G, P^g \leq Q\}.$$

Saturated fusion system

Definition

A *saturated* fusion system over a finite p -subgroup S is the category $\mathcal{F}$ whose objects are the subgroups of S and has morphism sets

$$\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) \subseteq \mathrm{Inj}(P, Q),$$

that satisfy additional axioms so that S “behaves” as a Sylow p -subgroup of $\mathcal{F}$.

Realizability

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Proposition ([Lib08])

A group which realizes a saturated fusion system is pseudo finite at the prime p .

$\mathcal{F}$ -collections

Definition

Fix a fusion system $\mathcal{F}$ over a finite p -group S . An $\mathcal{F}$ -collection is a union of $\mathcal{F}$ -conjugacy classes of subgroups of S . An $\mathcal{F}$ -collection $\mathcal{C}$ is *closed* if a subgroup $Q \leq S$ belongs to $\mathcal{C}$ whenever it contains an element of $\mathcal{C}$.

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if and only if there is some isomorphism $\varphi \in \text{Hom}_{\mathcal{F}}(P_n, Q_n)$ such that $\varphi(P_i) = Q_i$ for every $0 \leq i \leq n$.

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- We denote the quotient $N(\mathcal{C}) / \sim_{\mathcal{F}}$ as $N(\mathcal{C}) / \mathcal{F}$.

Main Result

Theorem (D., Villareal)

Let $\mathcal{F}$ be a saturated fusion system over the finite p -group S and let $\mathcal{C}$ be a closed $\mathcal{F}$ -collection. The simplicial set $N(\mathcal{C})/\mathcal{F}$ admits a Morse matching where the set of critical cells consists of one cell of dimension 0, and hence $N(\mathcal{C})/\mathcal{F}$ is contractible.

Proof I

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- $\mathcal{F} = \mathcal{F}_S(G)$ where G is some pseudo finite group at the prime p and $S \in \text{Syl}_p(G)$.

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- Define $\hat{\mathcal{C}} = \bigcup_{P \in \mathcal{C}} P^G$, where P^G denotes the G -conjugacy class of P .

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- Since $\hat{\mathcal{C}}$ is a collection of finite p -subgroups of G that is closed under conjugation and passage to overgroups in $\mathcal{S}_p(G)$, $N(\hat{\mathcal{C}})/G$ is contractible.

Proof II

Proposition ([Lib08])

Let G be a group that realizes the saturated fusion system $\mathcal{F}$ on the finite p -group S . Let $\mathcal{C}$ be an $\mathcal{F}$ -collection. Then $N(\mathcal{C})/\mathcal{F} \rightarrow N(\widehat{\mathcal{C}})/G$ is an isomorphism of simplicial sets.

Proof II

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Hence, $N(\mathcal{C})/\mathcal{F}$ is contractible.

Thank you for your attention!



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Questions?