A Nerve Theorem for Cyclic Multicategories

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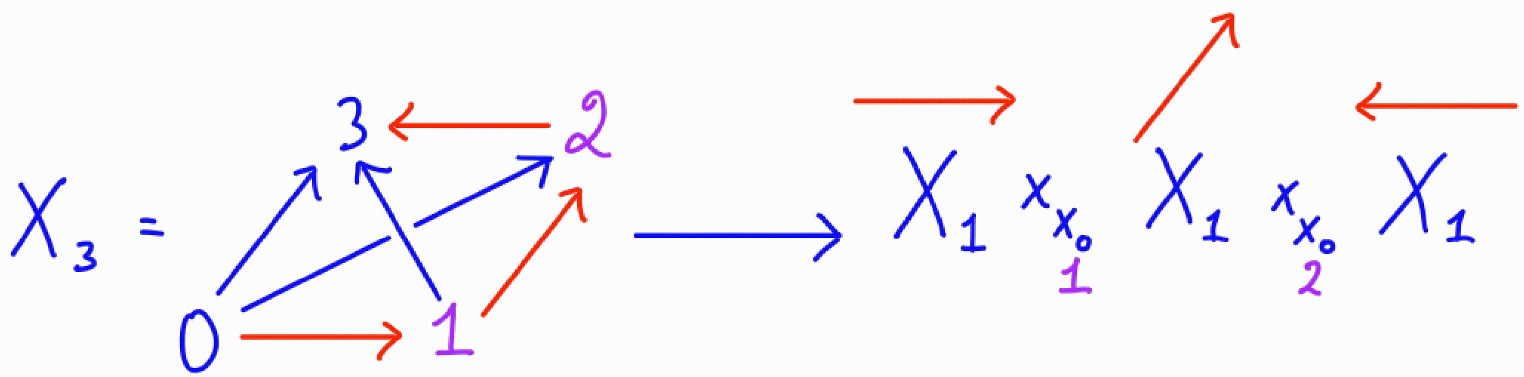
THE Nerve Theorem

Theorem: Let X be a simplicial set. Then $X \cong N\mathcal{C}$ for some category $\mathcal{C}$ if and only if X satisfies the Segal condition, namely for all $n \geqslant 2$ the map sending n-simplices of X to its spine is a bijection $\boxed{X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1}$.



Goal:

↳ Find analogous nerve theorems for cyclic multicategories and operads.

↳ Namely, characterize cyclic multicategories as presheaves on the category of planar trees.

↳ Using machinery developed by Berger, Melliès, and Weber.

Definition: Given an inclusion of a full subcategory $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{E}$, the associated nerve functor $\nu_{\mathcal{A}} : \mathcal{E} \to \widehat{\mathcal{A}}$ is defined as follows: for each object X of $\mathcal{E}$, $\nu_{\mathcal{A}}(X)$ is given by $\boxed{\nu_{\mathcal{A}}(X)(A) = \mathcal{E}(i_{\mathcal{A}}(A), X)}$, where A is an object of $\mathcal{A}$.
$\mathcal{A}$ is called a dense generator of $\mathcal{E}$ if $\mathcal{A}$ is small and $\nu_{\mathcal{A}}$ is fully faithful.

Example: $[3] \in \mathcal{A}$: $\bullet \to \bullet \to \bullet \to \bullet$

↳ $\mathcal{E} = \mathrm{DirGra}$, $\mathcal{A} = \Delta_{\mathrm{inert}} \Rightarrow \mathcal{N}_{\mathcal{A}}(G)([n])$ is the set of length n paths in G.

Δ but only with subinterval inclusions ("distance preserving")

↳ $\mathcal{E} = \mathrm{Cat}$, $\mathcal{A} = \Delta \Rightarrow \mathcal{N}_{\mathcal{A}}(\mathcal{C})([n])$ is the set of n composible morphisms in $\mathcal{C}$.

Definition: Let $\mathcal{C}$ be a category and let $T : \mathcal{C} \to \mathcal{C}$. A **monad on $\mathcal{C}$** is a triple (T, η, μ) that is a monoid object in the category of endofunctors of $\mathcal{C}$.

Remark: $L : \mathcal{E} \rightleftarrows \mathcal{D} : R$ adjunction $\rightsquigarrow T = RL$ monad on $\mathcal{E}$.

(η is the unit of the adjunction and $\mu = R\varepsilon L$, where ε is the counit)

Definition: A **T-algebra** (x, h) is an object x of $\mathcal{E}$, together with a morphism $h : Tx \to x$ in $\mathcal{E}$ satisfying compatibility (μ) and unit (η) axioms.

The category of T-algebras is called the Eilenberg-Moore category and denoted $\mathcal{E}^T$.

Example: $L : \mathrm{DirGra} \rightleftarrows \mathrm{Cat} : R$

Free category   Underlying directed graph

$\rightsquigarrow$ $T = RL$ monad on DirGra and $\mathrm{DirGra}^T = \mathrm{Cat}$.

Leinster-Weber Theory

Definition: A monad T on a category $\mathcal{E}$ with dense generator $\mathcal{A}$ is called a monad with arities $\mathcal{A}$ if the composite $\nu_{\mathcal{A}} T$

takes $\mathcal{A}$-cocones in $\mathcal{E}$ to colimit-cocones in $\widehat{\mathcal{A}}$.

Berger, Mellies, Weber – Monads with arities and their associated theories

Theorem: (BMW '12, Theorem 1.10) Let $\mathcal{E}$ be a category with dense generator $\mathcal{A}$. If T is a monad on $\mathcal{E}$ with arities $\mathcal{A}$, let Θ_T be the full subcategory of $\mathcal{E}^T$ spanned by the free T-algebras on the arities and $\nu_T : \mathcal{E}^T \longrightarrow \widehat{\Theta_T}$ is the associated nerve functor. Then the following hold:

- Θ_T is a dense generator for $\mathcal{E}^T$, i.e. ν_T is fully faithful.
- The essential image of ν_T is spanned by those presheaves whose restriction along $j_T : \mathcal{A} \to \Theta_T$ belongs to the essential image of $\nu_{\mathcal{A}}$.

"generalized" Segal condition

Example: Let T be the free category monad.

$\mathcal{A} = \Delta_{inert} \subseteq DirGra \rightsquigarrow \Theta_T = \Delta \subseteq Cat.$

↳ $N : Cat \longrightarrow sSet$ is fully faithful.

↳ $X \in sSet$, $X \cong N\mathcal{C}$ for some category $\mathcal{C}$

$\iff j_T^* X$ is in the essential image of ν_4

$\iff \exists G \in DirGra$ such that $j_T^* X \cong \nu_4 G$

$\iff \forall [n] \in \Delta_{inert}$, $X_n \cong$ paths of lenghs n in G

$\iff \forall [n] \in \Delta_{inert}$, $X_n \cong E(G) \times_{V(G)} \cdots \times_{V(G)} E(G)$

$\iff \forall n \geqslant 2$, $X_n \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$

Definition: A functor $R : \mathcal{D} \to \mathcal{E}$ is said to be a local right adjoint if for each object $d \in \mathcal{D}$, $R_d : \mathcal{D}/d \longrightarrow \mathcal{E}/Rd$ given by

$$\begin{pmatrix} d' \\ \downarrow f \\ d \end{pmatrix} \mapsto \begin{pmatrix} Rd' \\ \downarrow R(f) \\ Rd \end{pmatrix}$$

admits a left adjoint $L_d : \mathcal{E}/Rd \longrightarrow \mathcal{D}/d$.

Lemma: (BMW '12, Lemma 2.7) Assume $\mathcal{D}$ has a terminal object 1. A functor $R: \mathcal{D} \to \mathcal{E}$ is a local right adjoint if and only if the induced functor $R_1: \mathcal{D} \to \mathcal{E}/R1$ has a left adjoint $L_1: \mathcal{E}/R1 \to \mathcal{D}$.

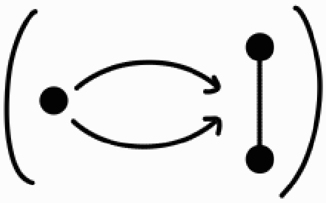
Definition: Let T be a monad on a category $\mathcal{E}$ with pullbacks. If T preserves pullbacks and if all naturality squares of η and μ of T are pullbacks, we say that T is cartesian.
A cartesian monad is called strongly cartesian if T is a local right adjoint.

Theorem: (BMW '12, Theorem 2.9) Let T be a strongly cartesian monad on a finitely complete category $\mathcal{E}$.

Any dense generator $\mathcal{A}$ of $\mathcal{E}$ embeds in a dense generator $\mathcal{A}_T$ such that T has arities $\mathcal{A}_T$.

↳ <u>Construction</u>: $\mathcal{A}_T = \left\{ L_1\left(\begin{smallmatrix} A \\ \downarrow \\ T1 \end{smallmatrix}\right) \;\middle|\; \begin{smallmatrix} A \\ \downarrow \\ T1 \end{smallmatrix},\ A \in \mathcal{A} \right\}$

Left adjoint of $T_1 : \mathcal{E} \to \mathcal{E}/_{T1}$

<u>Example</u>: $\mathbb{D} =$  , $\mathrm{DirGra} = \widehat{\mathbb{D}}$

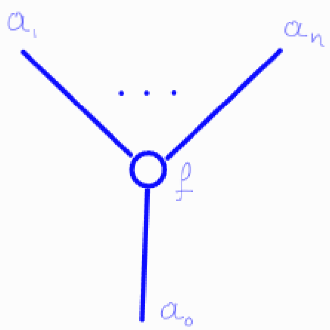
can't build paths of lenght ≥ 2

Representables $\mathcal{A} = \mathbb{D}$ is a dense generator but <u>NOT</u> arities for T, the free category monad $\overset{\text{Construction}}{\rightsquigarrow}$ $\mathcal{A}_T = \Delta_{inert}$.

<u>Definition</u>: A <u>multicategory</u> $\mathcal{M}$ consists of the following data:

↳ A set of <u>colors</u> $Col(\mathcal{M})$, and

↳ $\forall n \geq 0$ and colors $a_0, \dots, a_n \in Col(\mathcal{M})$, a set $\mathcal{M}(a_1, \dots, a_n; a_0)$ of <u>n-ary operations</u> equipped with

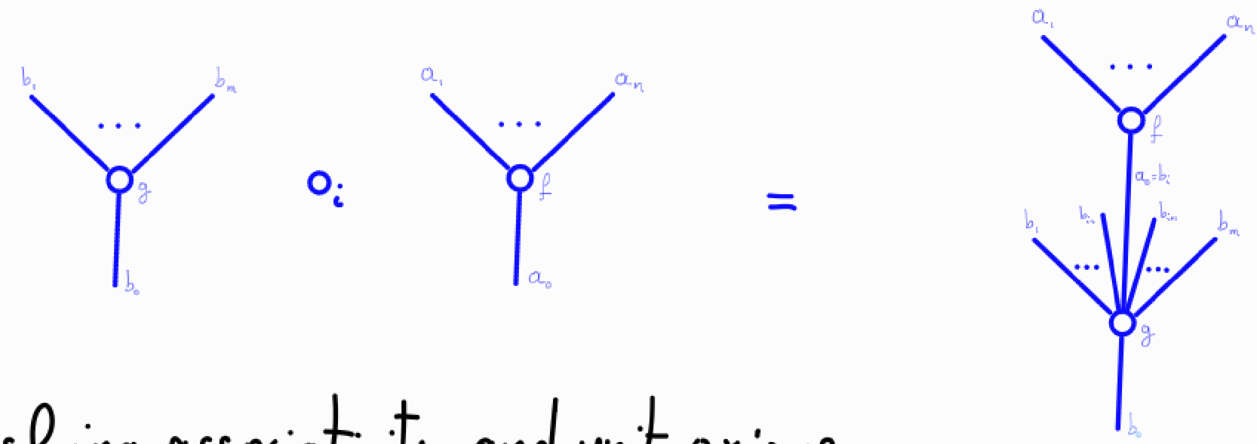


↳ <u>identities:</u> $\forall a \in Col(\mathcal{M})$, $id_a \in \mathcal{M}(a;a)$

↳ <u>composition:</u> $\forall n,m \geq 0$ and colors $a_0,\dots,a_n,b_0,\dots,b_m \in Col(\mathcal{M})$

if $a_0 = b_i$ for some $1 \leq i \leq m$, a function

$$\circ_i : \mathcal{M}(b_1,\dots,b_m;b_0) \times \mathcal{M}(a_1,\dots,a_n;a_0) \longrightarrow \mathcal{M}(b_1,\dots,b_{i-1},a_1,\dots,a_n,b_{i+1},\dots,b_m;b_0)$$



satisfying associativity and unit axioms.

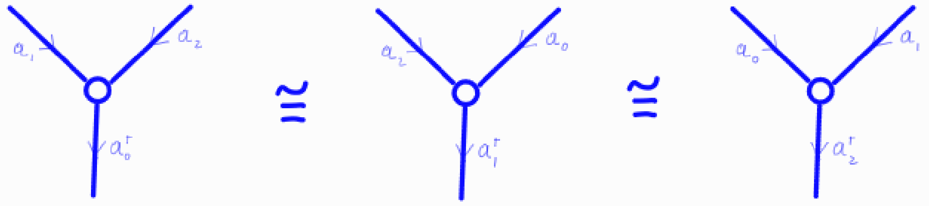
Cheng, Gurski, Riehl - Cyclic multicategories, multivariable adjunctions and mates

<u>Definition:</u> (CGR '14, Definition 3.3) A multicategory $\mathcal{M}$ is <u>cyclic</u> if it is equipped with

↳ An involution on colors $a \longmapsto a^\dagger$, and

↳ $\forall n \geq 1$ and colors $a_0,\dots,a_n$ an isomorphism

$$\sigma = \sigma_{n+1} : \mathcal{M}(a_1,\dots,a_n;a_0^\dagger) \xrightarrow{\cong} \mathcal{M}(a_2,\dots,a_n,a_0;a_1^\dagger)$$



satisfying the following:

- Each isomorphism σ_n is so that $\sigma_n{}^n = 1$.
- Identities are preserved by σ_2, i.e. $\sigma_2 \, id_a = id_{a^\dagger}$.
- Interaction between σ and composition, namely

for all compatible n-ary operation f and m-ary operation g,

$$\sigma(g \circ_i f) = \begin{cases} \sigma f \circ_n \sigma g, & i = 1 \\ \sigma g \circ_{i-1} f, & 2 \leq i \leq m \end{cases}$$

planar (unrooted) trees + CMC maps between them

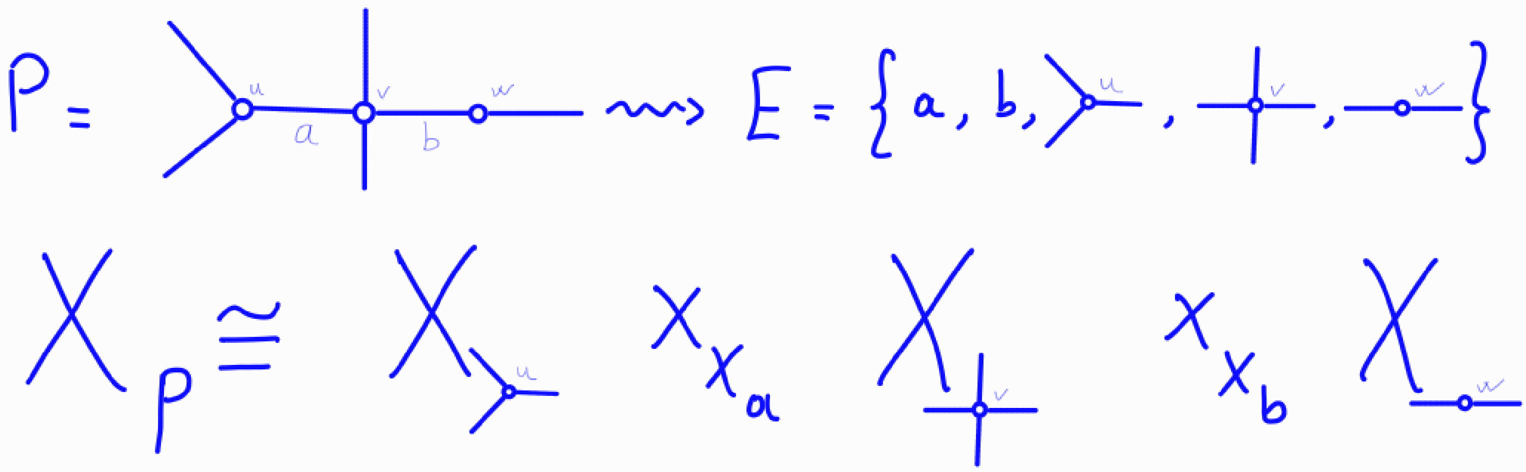
"cyclic dendroidal sets"

Theorem: (D.) Let $\nu : CMC \longrightarrow \widehat{\mathrm{Tree}}$ be the nerve functor associated to the full inclusion $\mathrm{Tree} \hookrightarrow CMC$.

Then the following holds:

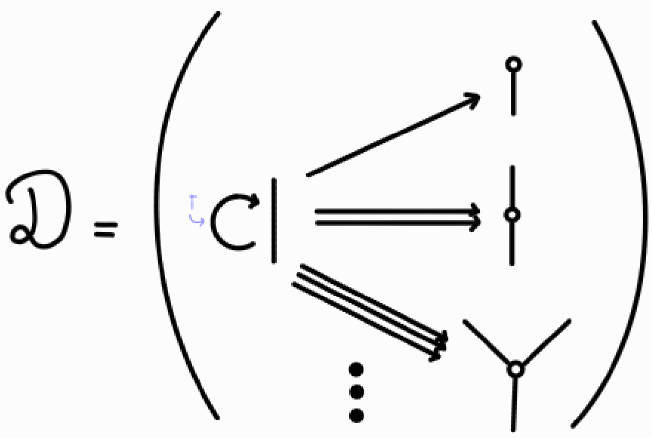
- The nerve functor ν is fully faithful.

↳ If $X \in \widehat{\mathrm{Tree}}$, then X is the nerve of some cyclic multicategory $\mathcal{M} \in \mathrm{CMC}$ if and only if for every tree $P \in \mathrm{Tree}$, $X_P \cong \lim_E X_E$ where E consists of the internal edges of P, , and stars, , for each vertex.



Proof (outline):

↳ Consider the diagram category:



↳ Define $i\text{-}MG := \widehat{\mathcal{D}}$ with dense generator the representables $\mathcal{A} = \mathcal{D}$.

↳ Construct an adjunction $L : i\text{-}MG \rightleftarrows CMC : R$

Free cyclic multicategory — Underlying involutive multigraph

↳ colors: G_e

↳ identities: $\hat{E} \to G$

↳ operations: $\hat{P} \to G$

⇝ to get a monad T on $i\text{-}MG$.

↳ Show T is strongly cartesian and since $i\text{-}MG := \widehat{\mathcal{D}}$,

i-MG is finitely complete

we apply the arities construction wich yields $\mathcal{A}_T = \mathrm{Tree}_{\text{inert}}$

Tree with only subtree inclusions

⇝ free T-algebras $\Theta_T = \mathrm{Tree}$.

↳ Apply BMW theorem and contextualize meaning.

Remark: Have analogous result for multigraphs/multicategories with arities planar rooted trees.

not cyclic

Future work:

↳ Augmented cyclic multicategories: allow 0-ary operations with no output

$g \in \mathcal{M}(;a^+) \cong \mathcal{M}(a;) \quad \circ_1 \quad f \in \mathcal{M}(;a^+) \cong \mathcal{M}(a;) \quad = \quad g \circ_1 f \in \mathcal{M}(;)$

↳ Cyclic operads: give a new proof for the nerve theorem (first appearing in Patrick Elliott's thesis).

Thank You!